Knox Grammar School

2016

Trial Higher School Certificate Examination

Name: ____________________

Teacher: ____________________

Year 12 Extension 2 Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- The official BOSTES Reference Sheet is provided
- In Questions 11-16, show relevant mathematical reasoning and/or calculations.

Section I ~ Pages 3-6

- 10 marks
- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II ~ Pages 7-13

- 90 marks
- Attempt Questions 11-16
- Allow about 2 hours 45 minutes for this section

Teachers:

Mr Bradford (Examiner)
Ms Yun

Write your name on the front cover of each answer booklet

This paper MUST NOT be removed from the examination room.
Number of Students in Course: 26

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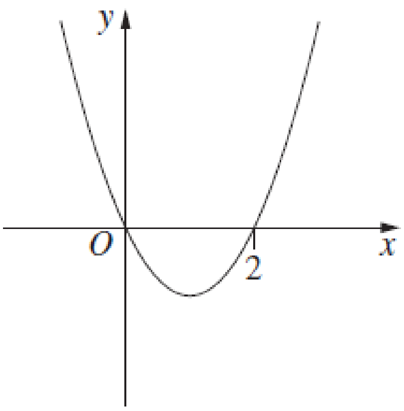
Section I

10 marks
Attempt questions 1 – 10
Allow about 15 minutes for this section

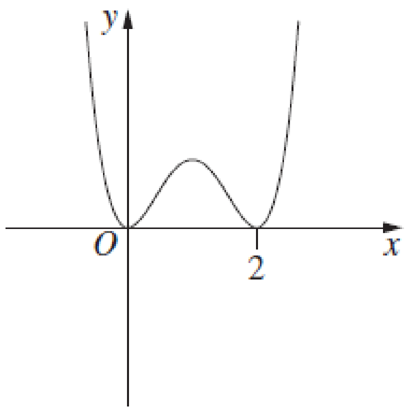
Use the multiple-choice answer sheet for Questions 1-10.

1 Which graph best represents the curve $y^2 = x^2 - 2x$.

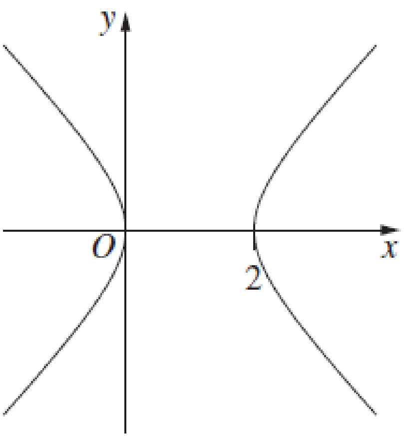
(A)



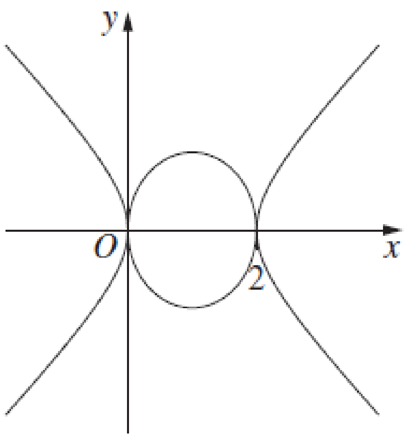
(B)



(C)



(D)



2 What value of z satisfies $z^2 = 7 - 24i$?

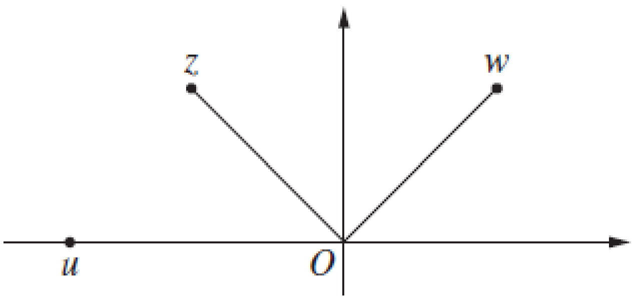
(A) $4 - 3i$

(B) $-4 - 3i$

(C) $3 - 4i$

(D) $-3 - 4i$

3 The Argand diagram shows the complex numbers w, z and u, where w lies in the first quadrant, z lies in the second quadrant and u lies on the negative real axis.



Which statement could be true?

(A) $u = zw$ and $u = z + w$

(B) $u = zw$ and $u = z - w$

(C) $z = uw$ and $u = z + w$

(D) $z = uw$ and $u = z - w$

4 If $y = x^2$ then $\int x\,dy =$

(A) $2y^{3/2} + C$

(B) $\frac{2}{3}x^{3/2} + C$

(C) $2y^3 + C$

(D) $\frac{2}{3}x^3 + C$

5 The angular speed of a disc of radius 5 cm is 10 revolutions per minute. What is the speed of a mark on the circumference of the disc?

(A) 50 cm min^{-1}

(B) $\frac{1}{2}$ cm min^{-1}

(C) 100π cm min^{-1}

(D) $\frac{1}{4\pi}$ cm min^{-1}

6 A particle is moving along a straight line so that initially its displacement is $x = 1$, its velocity is $v = 2$, and its acceleration is $a = 4$.
Which is a possible equation describing the motion of the particle?

(A) $v = 2\sin(x - 1) + 2$

(B) $v = 2 + 4\log_e x$

(C) $v^2 = 4(x^2 - 2)$

(D) $v = x^2 + 2x + 4$

7 The numbers 1, 2, . . . n, for $n \geq 4$, are randomly arranged in a row.
What is the probability that the number 1 is somewhere to the left of the number 2?

(A) $\frac{1}{2}$

(B) $\frac{1}{n}$

(C) $\frac{1}{2(n-2)!}$

(D) $\frac{1}{2(n-1)!}$

8 A hostel has four vacant rooms. Each room can accommodate a maximum of four people. In how many different ways can six people be accommodated in the four rooms?

(A) 4020

(B) 4068

(C) 4080

(D) 4096

9 Which expression is equal to $\int \frac{1}{1-\sin x}\,dx$?

(A) $\tan x - \sec x + c$

(B) $\tan x + \sec x + c$

(C) $\log_e(1 - \sin x) + c$

(D) $\frac{\log_e(1-\sin x)}{-\cos x} + c$

10 Which integral is necessarily equal to $\int_{-a}^{a} f(x)\,dx$?

(A) $\int_0^a f(x) - f(-x)\,dx$

(B) $\int_0^a f(x) - f(a-x)\,dx$

(C) $\int_0^a f(x-a) + f(-x)\,dx$

(D) $\int_0^a f(x-a) + f(a-x)\,dx$

Section II

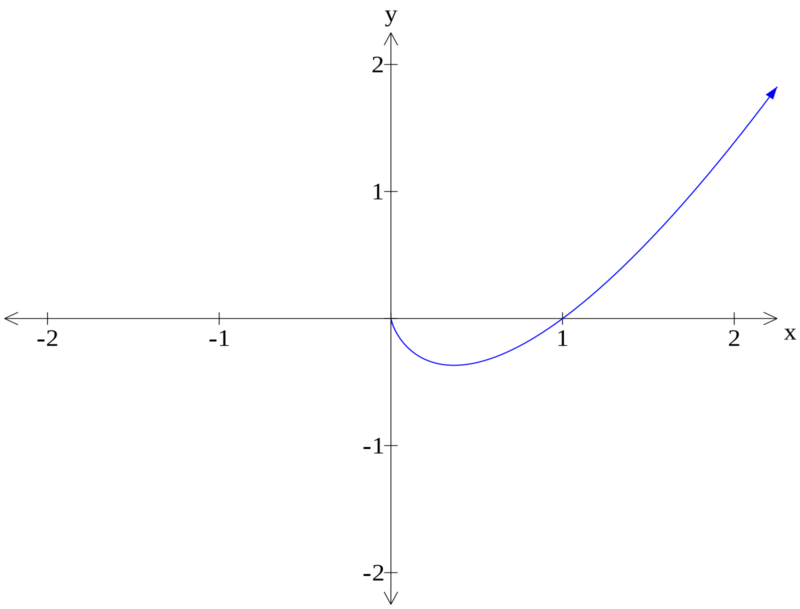
90 marks
Attempt questions 11 – 16
Allow about 2 hours 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

All necessary working should be shown in every question.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The diagram shows the graph of the function $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = -f(x)$ **1**

(ii) $y = |f(x)|$ **1**

(iii) $y = \dfrac{1}{f(x)}$ **2**

(iv) $y = (f(x))^2$ **2**

Question 11 continues on page 8

Question 11 (continued)

(b) (i) Sketch the region in the complex plane where the inequalities $|z-i|\le 2$ and $0\le Arg(z-1)\le \frac{3\pi}{4}$ hold simultaneously. **3**

(ii) What is the minimum value of $|z|$? **2**

(iii) What is the value of $Arg\left(|z|\right)$? Give reasons for your answer. **1**

(iv) What is the maximum value of $Arg\left(z\right)$?
Give your answer in radians, correct to two decimal places. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Find real numbers *a*, *b* and *c* such that **3**

$$\frac{7x+4}{\left(x^2+1\right)\left(x+2\right)}=\frac{ax+b}{x^2+1}+\frac{c}{x+2}$$

(ii) Hence find $\int \frac{7x+4}{\left(x^2+1\right)(x+2)}dx$ **2**

(b) Let $\alpha=-\sqrt{3}+i$.

(i) Express α in modulus-argument form. **2**

(ii) Hence find the least positive integer *n* for which α^n is purely real. **2**

(c) By taking slices perpendicular to the axis of rotation, find the volume of the solid formed when the region bounded by the curves $y=2x^3$ and $y=2\sqrt{x}$ is rotated about the *x*-axis. Explain your reasoning carefully by sketching the curves and using sigma notation. **4**

(d) The expression $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+...}}}}$ has a limit *L*. Find the exact value of *L*. Explain your reasoning carefully. **2**

Question 13 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Let α, β and γ be roots of the equation $x^3+x^2-2x-5=0$. For the following questions, give your answers in the form $ax^3+bx^2+cx+d=0$ where the coefficients *a*, *b*, *c* and *d* are integers.

(i) Find a polynomial equation with integer coefficients whose roots are $\alpha-2, \beta-2$ and $\gamma-2$. **2**

(ii) Find a polynomial equation with integer coefficients whose roots are α^2, β^2 and γ^2. **2**

(b) Solve the equation $x^4+x^2+6x+4=0$ over the complex field given that it has a rational zero of multiplicity 2. **4**

(c) Find the values of real numbers *p* and *q* such that $1-i$ is a root of the equation $z^3+pz+q=0$. **3**

(d) (i) Using the identity $(p+q)^2=(p-q)^2+4pq$, show that for $p, q>0$

$$\frac{(p+q)}{2}\geq\sqrt{pq}$$ **2**

(ii) Hence show that if *p*, *q*, *r* and *s* are greater than zero then

$$\frac{p+q+r+s}{4}\geq\sqrt[4]{pqrs}\,.$$ **2**

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Consider the hyperbola with the equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

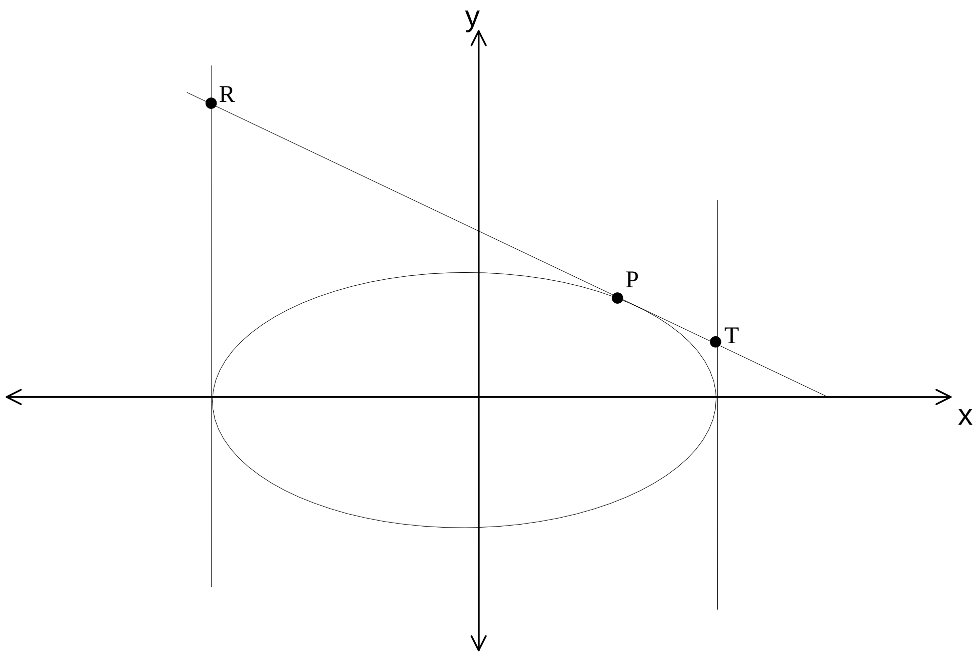
(i) What is the eccentricity of the hyperbola? **1**

(ii) Find the coordinates of the foci and *x*-intercepts of the hyperbola. **2**

(iii) Find the equations of the directrices and the asymptotes of the hyperbola. **2**

(iv) What are the parametric equations of this hyperbola? **1**

(b)



The point P lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$. The tangent at P meets the tangents at the ends of the major axis at R and T.

(i) Use the parametric representation of an ellipse to show the equation of the tangent is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$. **2**

(ii) Show that RT subtends a right angle at the focus $S(ae, 0)$. **3**

(c) By deriving the equation of the tangent, prove that the chord of contact of the tangents from the point $P(x_0, y_0)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has the equation $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$. **4**

Question 15 (15 marks) Use a SEPARATE writing booklet **Marks**

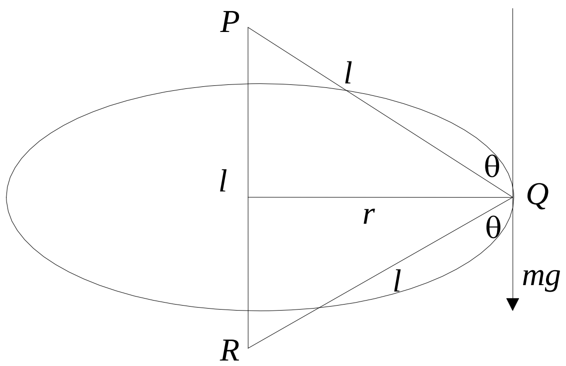
(a) A particle of mass m is moving in a straight line under the action of a force F, where:

$$F = \frac{m}{x^3}(6 - 10x).$$

(i) Find v^2 in any position if the particle starts from rest at $x = 1$. **2**

(ii) At which other position does the particle come to rest? **2**

(b) Two light inextensible strings PQ and QR each of length l are attached to a particle of mass m at Q. The other ends P and R are fixed to two points in a vertical line such that P is a distance l above R. The particle describes a horizontal circle with constant angular speed ω.



(i) Find the tension in the strings. **3**

(ii) What value must ω be greater than in order for the strings to be tight? **1**

(c) Find, correct to one decimal place, the angle at which a road must be banked so that a car may round a curve with a radius of 200 metres at 100 km/h without sliding. Use a diagram and appropriate force equations in your solution. **2**

(d) (i) Differentiate $\sin^{n-1}\theta\cos\theta$ expressing the result in terms of $\sin\theta$ only. **2**

(ii) Hence, or otherwise, deduce that if $I_n = \int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta$, then

$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$ with $n \geq 2$. **2**

(iii) Hence find the exact value of I_4. **1**

Question 16 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Prove the following relationship for $n \geq 2$ using mathematical induction.

$$n^{n+1} > n(n+1)^{n-1}$$ **3**

(b) Suppose that $x \geq 0$ and n is a positive integer.

(i) Show that $1 - x \leq \frac{1}{1+x} \leq 1$. **2**

(ii) Hence, or otherwise, show that $1 - \frac{1}{2n} \leq n \ln\left(1 + \frac{1}{n}\right) \leq 1$. **2**

(iii) Hence, explain why $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$. **1**

(c) On polling day the ratio of the electoral votes in the three available booths A, B and C was 5:4:3 respectively. The percentage of votes for Mr Turnbull in these booths was 30%, 60% and 50% respectively.

If ten voters were chosen at random, find the probability that Mr Turnbull gained at least eight votes. Give your answer correct to five decimal places. **3**

(d) (i) Prove that $(a+b+c)^2 \geq 3(ab+ac+bc)$.
Note that a, b and c are positive integers. **2**

(ii) Hence or otherwise prove that for positive integers x, y and z

$$x^2y^2 + x^2z^2 + y^2z^2 \geq xyz(x+y+z)$$ **2**

End of paper

2016 Year 12 Mathematics Extension 2 Task 5 Trial HSC SOLUTIONS

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
SECTION I - MULTIPLE CHOICE 1. For the relation to be defined, $y^2 \geq 0 \Rightarrow x^2 - 2x \geq 0$ $\Rightarrow x \leq 0$ or $x \geq 2$. Of the alternatives only (C) is therefore acceptable. Indeed:- $y^2 = x^2 - 2x \Rightarrow (x-1)^2 - y^2 = 1$ & we have the eqn of a rectangular hyperbola, centred at $(1, 0)$. 2. Easiest approach is direct substitution. Moreover, as the cross-term in the binomial exp. yields the term in i and it is negative, viz., $-24i$, then only (A) or (C) need be considered. By inspection (A) is the correct alternative. 3. Using the parallelogram of vectors approach, $u = z + w$ would place u on the positive imaginary axis. So only need consider (B) or (D). Also, mult^{n} of complex nos involves addition of their arguments, $z = uw$ would place z in the 3^{rd} quadrant, which is patently false. So (B) is the only viable alternative. 4. If $y = x^2$, then $dy = 2x\,dx$. $\Rightarrow \int x\,dy = \int 2x^2\,dx$	1.C 6.A 2.A 7.A 3.B 8.A 4.D 9.B 5.C 10.D	& $\int 2x^2\,dx = \frac{2}{3}x^3 + C$ So (D) 5. Each revolution is 10π cm. $\therefore$ 10 rev/min = 100π cm/min. So (C) 6. When $x = 1$, $v = 2$ & consequently only (A) & (B) are viable. Also, $a = 4$. Now $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ or $a = v \cdot \frac{dv}{dx}$. Using the latter, for convenience:- $a = \{2\sin(x-1) + 2\} \cdot 2\cos(x-1)$ $= 4$ when $x = 1$. Appears that (A) is the correct alternative. For confirmation, note that $a = (2 + 4\log_e x) \cdot \frac{4}{x}$ for (B) with $a = 8$ when $x = 1$. So (A) 7. "Common sense" would dictate that 1 is eith to the left of 2 or equally likely to be to the right of 2 & so the prob. should be $\frac{1}{2}$. Formally:- $\dfrac{1 \times (n-2)! + 2 \times (n-2)! + 3 \times (n-2)! + \cdots + (n-1) \times (n-2)!}{n!}$ $= \dfrac{(n-2)!\{1 + 2 + \cdots + (n-1)\}}{n!}$ $= \dfrac{(n-2)!\,\frac{1}{2}n(n-1)}{n!}$ or $\frac{1}{2}\frac{n!}{n!}$ $= \frac{1}{2}$, as expected. So (A)	Can you work out the logic?

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

8. Easier to work with complementary events. With no restriction there are 4^6 choices. We need to subtract from this value the no. of arrangements where all people are in the one room. Additionally, need to subtract all instances where five people are in one room with one individual in another. For the former, this is achieved in ${}^4C_1$ ways (why?). For the latter, this is achieved in ${}^4C_1 \times {}^6C_1 \times {}^3C_1$

- ${}^4C_1$: Choose a room for one person
- ${}^6C_1$: Select the individual
- ${}^3C_1$: Choose remaining room for five remaining people.

All up, $4^6 - {}^4C_1 - {}^4C_1 \times {}^6C_1 \times {}^3C_1$

$= 4020.$ So (A)

9. $\dfrac{1}{1-\sin x} = \dfrac{1}{1-\sin x} \cdot \dfrac{1+\sin x}{1+\sin x}$

$= \dfrac{1+\sin x}{\cos^2 x}$

$= \sec^2 x + \sec x . \tan x$

$\therefore \displaystyle\int \frac{dx}{1-\sin x} = \tan x + \sec x + C$

Comments: t substitution available. So (B)

10. $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

with the latter equal to $\displaystyle\int_0^a f(a-x)\,dx$ (Core theory)

Now $\displaystyle\int_0^a (f(x) - f(a-x))\,dx$ is necessarily equal to zero & so this alternative is discounted. So consider (D).

For $\displaystyle\int_0^a f(x-a)\,dx$, let $u = x-a \Rightarrow du = dx$

$\Rightarrow u = -a$ when $x = 0$ &

$u = 0$ when $x = a$.

Accordingly, integral becomes:

$\displaystyle\int_{-a}^{0} f(u)\,du$ or $\displaystyle\int_{-a}^{0} f(x)\,dx$

So (D) is correct!

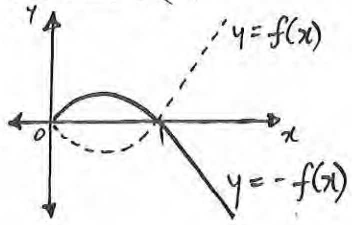
Comments: You could test the alternatives against a basic, non-trivial f^n, such as $f(x) = e^x$.

A student should anticipate that the core 4U Theory, viz., $\displaystyle\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$ is being tested here.

Section II

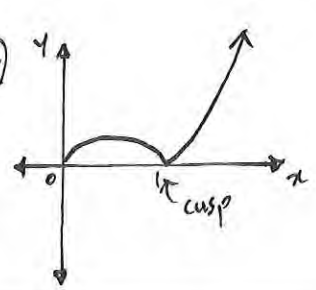
Question 11: (a)

i)



Comments: 1 mark

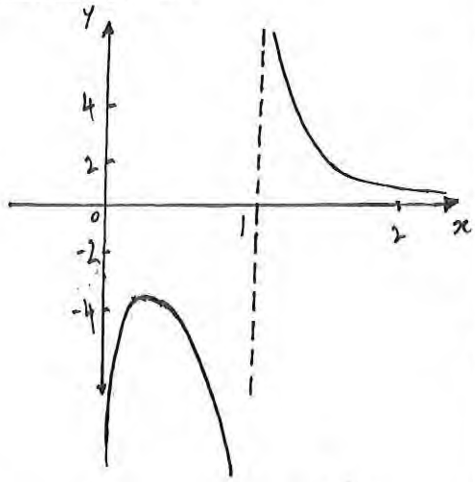
ii)



Comments: 1 mark

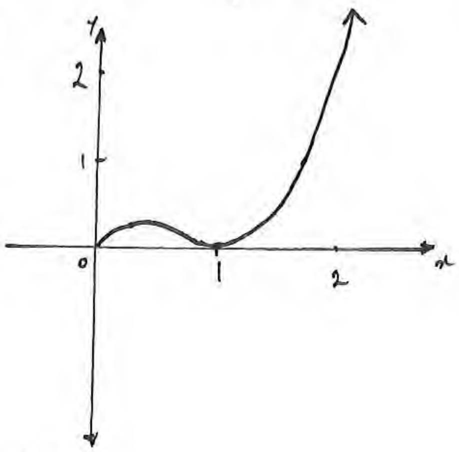
Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Q11 Cont. (a) iii)



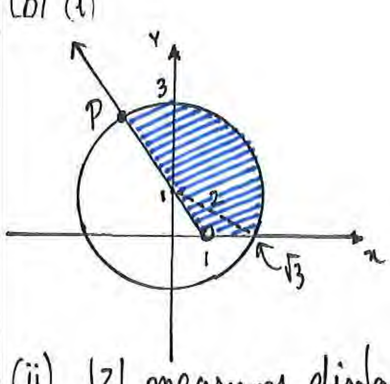
Comments: y coordinate of turning pt < -2. ✓ for shape. ✓ vertical asymptotes $x = 0, 1$.

iv)



Comments: ✓ for shape. ✓ turning pt at $x = 1$.

(b) (i)



Comments: ✓ shapes. ✓ shading. ✓ singularity.

(ii) $|z|$ measures distance from O. $|z|$ is minimised by calculating the perpendicular distance from the line through $(0,1)$ & $(1,0)$ to the origin.

$d_\perp = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$

Comments: Eq'n of line is $x + y - 1 = 0$. $d_\perp = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$, $(x_1, y_1) = (0,0)$, $a = b = 1; c = -1$

✓✓ correct answer with working
✓ Meaningful progress.

(iii) As the locus of z excludes the origin then $|z| > 0$ & so $Arg(|z|) = 0$ (a positive real no.)

Comments: ✓ for answer + reason; no credit for answer only.

(iv) $Arg(z)$ is maximised at the point P (see diagram) where the line $x + y - 1 = 0$ intersects $x^2 + (y-1)^2 = 4$ in the 2^{ND} quadrant.

To find P, solve simultaneously:

$$\begin{cases} x^2 + (y-1)^2 = 4 \\ x + y - 1 = 0 \text{ or } y - 1 = -x \end{cases}$$

Sub. $y - 1 = -x$ into eq'n of circle gives: $2x^2 = 4 \Rightarrow x = -\sqrt{2}$

$\Rightarrow P(-\sqrt{2}, \sqrt{2}+1)$.

$m_{OP} = \frac{\sqrt{2}+1}{-\sqrt{2}}$ or $-\frac{\sqrt{2}}{2} - 1$

& $Arg\left[-\sqrt{2} + i(\sqrt{2}+1)\right]$

$= 2{\cdot}10$ radians, correct to 2d.p.

Comments: iv) ✓✓✓ for correct sol'n plus working. ✓✓ Substantial progress. ✓ Recognition that Arg is max. at P, $x < 0$ for 2^{ND} quad. $2{\cdot}100\,699\ldots$

Question 12:

(a) (i) Equivalent to finding a, b & c when $7x + 4 \equiv (ax+b)(x+2) + c(x^2+1)$

When $x = -2 \Rightarrow -10 = 5c$ or $c = -2$

Equating coefficients of x^2: $a + c = 0$

$\Rightarrow a = 2$.

Equating constant terms: $2b + c = 4$

$\Rightarrow b = 3$

Comments: 1 mark for each correct ans. $a = 2$, $b = 3$, $c = -2$

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Suggested Solution (s)

Q12 cont.

a) ii)

$$\int \frac{7x+4}{(x^2+1)(x+2)}\,dx = \int\left(\frac{2x+3}{x^2+1} - \frac{2}{x+2}\right)dx$$

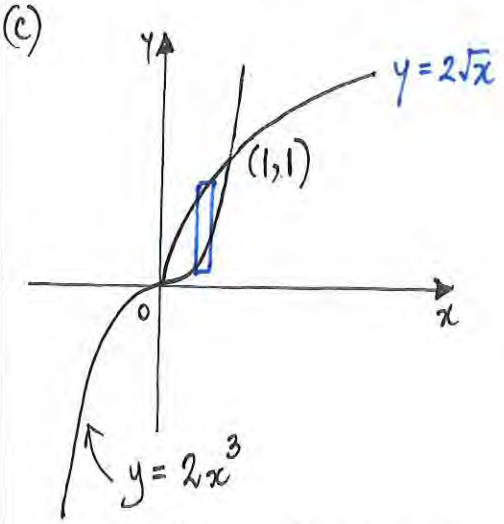
$$= \ln(x^2+1) + 3\tan^{-1}(x) - 2\ln|x+2| + C$$

(b) i) $\alpha = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

ii) Require least positive n for which $\sin\frac{5n\pi}{6} = 0$

$\Rightarrow n = 6$ by inspection

(c)



Divide the enclosed region into n rectangles of width δx. When a representative rectangle is rotated about the x-axis, it generates a solid whose base is an annulus with area A, where

$A = \pi y_2^2 - \pi y_1^2$. $\begin{cases} y_1 = 2x^3 \\ y_2 = 2\sqrt{x}. \end{cases}$

Comments

a)(ii) ✓✓ Correct ans. ✓ for meaningful progress

i) ✓ modulus ✓ argument.

ii) De Moivre's theorem employed. Formal methods available ✓✓ for Correct ans. + working ✓ meaningful progress.

(See diagram)

Suggested Solution (s)

$A = \pi(4x) - \pi(4x^6)$

$= 4\pi(x - x^6)$.

$\therefore V = \lim\limits_{\delta x \to 0} A(x)\cdot\delta x$

$= \lim\limits_{\delta x \to 0} 4\pi(x - x^6)\,\delta x$

$= 4\pi\int_0^1 (x - x^6)\,dx$

$= 4\pi\left[\frac{x^2}{2} - \frac{x^7}{7}\right]_0^1$

$= 4\pi\left(\frac{1}{2} - \frac{1}{7}\right)$

$= \frac{10\pi}{7}$ units^3

d) $L = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$

$\Rightarrow L^2 = 2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$

$\Rightarrow L^2 = 2 + L$

$\Rightarrow L^2 - L - 2 = 0$

$\Rightarrow L = 2,\ L > 0$.

Comments

1 mark: Determined a slice is an annulus; equivalently, volume of each section is a 'washer'.

2 marks: Correctly calculated area of annulus.

3 marks: Correct expression for V, namely $\lim\limits_{\delta x \to 0} 4\pi(x - x^6)\delta x$

4 marks: Correct solution

d) ✓✓ Correct answer plus working / Solution. ✓ for ans. alone

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Question 13:

(a)(i) Let $x = \alpha - 2 \Rightarrow \alpha = x+2$

α satisfies $x^3+x^2-2x-5=0$

$\Rightarrow (x+2)^3+(x+2)^2-2(x+2)-5=0$

$\Rightarrow x^3+6x^2+12x+8+x^2+4x+4-2x-9=0$

$\Rightarrow x^3+7x^2+14x+3=0$ ✓✓

ii) Let $x = \alpha^2 \Rightarrow \alpha = \pm\sqrt{x}$.

For convenience, chose $\alpha = \sqrt{x}$.

α satisfies $x^3+x^2-2x-5=0$

$\Rightarrow x\sqrt{x}+x-2\sqrt{x}-5=0$

$\Rightarrow \sqrt{x}(x-2) = 5-x$

$\Rightarrow x(x-2)^2 = (5-x)^2$

$\Rightarrow x^3-4x^2+4x = 25-10x+x^2$

$\Rightarrow x^3-5x^2+14x-25=0$ ✓✓

(b) Let $P(x) = x^4+x^2+6x+4$

$\Rightarrow P'(x) = 4x^3+2x+6$

Now $P'(-1)=0$ by inspection. Additionally, $P(-1)=0$. Conclude from the multiple root theorem that -1 is a root of multiplicity 2. It is also rational.

So $x^4+x^2+6x+4 = (x^2+2x+1)\times(ax^2+bx+c)$ for real nos. a, b & c. By inspection $a=1$ & $c=4$. Looking at the co-efficient of x^3: $0 = b+2a$

$\Rightarrow b=-2$.

Comment: Various methods available to find remaining quad. factor

So $x^4+x^2+6x+4 \equiv (x+1)^2(x^2-2x+4)$.

For the latter quadratic factor:

$x^2-2x+4=0 \Rightarrow (x-1)^2-(\sqrt{3}i)^2=0$

$\Rightarrow x = 1\pm i\sqrt{3}$

$\therefore$ roots are $-1\,(2)$, $1\pm i\sqrt{3}$

(c) From Conjugate root theorem, $1+i$ is also a root of cubic eq'n.

Let α be remaining root.

$\sum$ roots $= 0 \Rightarrow (1+i)+(1-i)+\alpha=0$

$\Rightarrow \alpha=-2$

So cubic is $(x-(1+i))(x-(1-i))(x+2)$

$\Rightarrow (x+2)(x^2-2x+2)=0$

$\Rightarrow x^3-2x+4=0$ whence

$p=-2$ & $q=4$

(d)(i) As $p, q > 0$, then $(p+q)^2 = (p-q)^2+4pq$ implies $(p+q)^2 \geq 4pq$ as $(p-q)^2 \geq 0$

$\Rightarrow p+q \geq 2\sqrt{pq}$ upon taking the positive sq. root

or $\frac{p+q}{2} \geq \sqrt{pq}$ as required

(ii) $\frac{p+q+r+s}{4} = \frac{\frac{p+q}{2}+\frac{r+s}{2}}{2}$

$\geq \sqrt{\left(\frac{p+q}{2}\right)\left(\frac{r+s}{2}\right)}$ using (i)

$\geq \sqrt{\sqrt{pq}\cdot\sqrt{rs}}$ using (i) again

$= \sqrt[4]{pqrs}$

Comments:

(b) Comments
Completing the square.
1 mark: Calculates derivative and finds its zero.
2 marks: Recognises $(x+1)^2$ as a factor of polynomial.
✓✓✓ Correct solution.

(c) Comments
1 mark: Recognises $1+i$ as another root via Conjugate root thm
2 marks Recognition of $\alpha=-2$ as remaining root.
3 marks: Correct solution.

d(i) 2 marks for process
(ii) 2 marks for process

2016 Year 12 Mathematics Extension 2 Task 5 Trial HSC SOLUTIONS

Suggested Solution (s)	Comments
Question 14: a) (i)	
$e^2 = 1 + \frac{b^2}{a^2}$; $a^2 = 16$ & $b^2 = 9$.	$b^2 = a^2(e^2 - 1)$ $a = 4$ $b = 3$
$\Rightarrow e^2 = 1 + \frac{9}{16}$ or $\frac{25}{16}$	
$\Rightarrow e = 5/4 \quad (e > 0)$	
ii) foci are $(\pm ae, 0) \Rightarrow (\pm 5, 0)$	
& x intercepts are ± 4.	Accept $(\pm 4, 0)$.
iii) $x = \pm \frac{a}{e} \Rightarrow x = \pm \frac{16}{5}$	
Equations of asymptotes are $y = \pm \frac{b}{a} x$ $\Rightarrow y = \pm \frac{3}{4} x$	$3x \pm 4y = 0$
iv) $\begin{cases} x = a\sec\theta \\ y = b\tan\theta \end{cases} \Rightarrow \begin{cases} x = 4\sec\theta \\ y = 3\tan\theta \end{cases}$	$\theta \neq \pm \frac{\pi}{2}$
(b) (i) $x = a\cos\theta$ & $y = b\sin\theta$	
$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ or $\frac{b\cos\theta}{-a\sin\theta}$.	
Eq'n of tangent is :-	$y - y_1 = m(x - x_1)$
$y - b\sin\theta = \frac{b\cos\theta}{-a\sin\theta}(x - a\cos\theta)$	
$\Rightarrow -ay\sin\theta + ab\sin^2\theta = bx\cos\theta - ab\cos^2\theta$	
$\Rightarrow bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$	$\sin^2\theta + \cos^2\theta = 1$
$\Rightarrow \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ upon division by ab.	$ab \neq 0$ √√ for process

Suggested Solution (s)	Comments
(ii) We need to find the co-ords. of R & T.	
When $x = a$: $\cos\theta + \frac{y}{b}\sin\theta = 1$	
$\Rightarrow y = \frac{b(1 - \cos\theta)}{\sin\theta}$	1 mark: Determined co-ords. at T & R.
$\therefore T\left(a, \frac{b(1 - \cos\theta)}{\sin\theta}\right)$.	
When $x = -a$: $-\cos\theta + \frac{y}{b}\sin\theta = 1$	2 marks Made progress in proving lines are $\perp$ at the focus.
$\Rightarrow y = \frac{b(1 + \cos\theta)}{\sin\theta}$	
$\therefore R\left(-a, \frac{b(1 + \cos\theta)}{\sin\theta}\right)$	
$m_{RS} = \frac{b(1 + \cos\theta)/\sin\theta - 0}{-a - ae}$	
$= \frac{b(1 + \cos\theta)}{-a(1 + e)\cdot\sin\theta}$ ...(I)	
$m_{TS} = \frac{b(1 - \cos\theta)/\sin\theta - 0}{a - ae}$	
$= \frac{b(1 - \cos\theta)}{a(1 - e)\sin\theta}$ ...(II)	
For RT to subtend a right-angle at S, we require:	
$m_{RS} \times m_{TS} = -1$	
$LHS = \frac{b^2\sin^2\theta}{-a^2(1 - e^2)\sin^2\theta}$	$1 - \cos^2\theta = \sin^2\theta$
$= \frac{b^2}{-b^2}$ or -1, as desired.	$b^2 = a^2(1 - e^2)$ for an ellipse

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Question 14 Cont. (c) Let $Q(x_1,y_1)$ & $R(x_2,y_2)$ be distinct points on the ellipse. We proceed to find the equation of the tangent at Q as follows: $\frac{d}{dx}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\frac{d}{dx}(1)$ $\Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$ $\Rightarrow \frac{dy}{dx}=-\frac{b^2x}{a^2y}$ $=-\frac{b^2x_1}{a^2y_1}$ at Q. Eq'n of tangent is therefore:- $y-y_1=\frac{-b^2x_1}{a^2y_1}(x-x_1)$ $\Rightarrow a^2y_1y-a^2y_1^2=-b^2x_1x+b^2x_1^2$ $\Rightarrow b^2x_1x+a^2y_1y=b^2x_1^2+a^2y_1^2$	✓ for derivation of tangents ✓✓✓ for complete proof.	$\frac{x_1x_0}{a^2}+\frac{y_1y_0}{b^2}=1$. This eq'n can be interpreted as follows: $Q(x_1,y_1)$ lies on the line $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$ Also, for the tangent at R to pass through P we require:- $\frac{x_2x_0}{a^2}+\frac{y_2y_0}{b^2}=1$. So $R(x_2,y_2)$ lies on the line $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$. As Q & R both satisfy $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$, the equation of the chord PQ must be $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$, as required.	
$\Rightarrow \frac{x_1x}{a^2}+\frac{y_1y}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$ $\Rightarrow \frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$ as Q lies on ellipse. Similarly, the equation of the tangent at $R(x_2,y_2)$ is $\frac{x_2x}{a^2}+\frac{y_2y}{b^2}=1$. For the tangent at Q to pass through P we require	$\div$ by $a^2b^2 \neq 0$	Question 15: (a) i) By Newton's 2nd Law:- $F=ma \Rightarrow ma=\frac{m}{x^3}(6-10x)$ & $a=v\cdot\frac{dv}{dx}$. Consequently, $v\cdot\frac{dv}{dx}=\frac{6-10x}{x^3}$ $\Rightarrow \int v\frac{dv}{dx}\,dx=\int(6x^{-3}-10x^{-2})\,dx$ $\Rightarrow \frac{1}{2}v^2=\frac{-3}{x^2}+\frac{10}{x}+C$ Sub $x=1$ & $v=0 \Rightarrow C=-7$. $\Rightarrow \frac{1}{2}v^2=\frac{-3}{x^2}+\frac{10}{x}-7$ or $v^2=-\frac{6}{x^2}+\frac{20}{x}-14$ ✓✓	$\int v\cdot\frac{dv}{dx}dx=\int v\,dv$. Equivalently, use $a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Question 15 cont.

a) (ii) Particle comes to rest when $v^2 = 0 \Rightarrow \frac{-6}{x^2} + \frac{20}{x} - 14 = 0$ — *Comment: (×) by x^2*

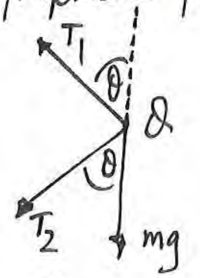
$\Rightarrow 14x^2 - 20x + 6 = 0$ or equivalently $7x^2 - 10x + 3 = 0$

We know $(x-1)$ is a factor

& so $7x^2 - 10x + 3 = (7x-3)(x-1)$ — *Comment: ✓✓*

$\therefore v^2 = 0$ when $x = \frac{3}{7}$ too.

(b) An appropriate force diagram is:



Resolving forces horizontally: — *Comment: Radially inward dir'n taken as posit.*

$T_1 \sin\theta + T_2 \sin\theta = m r \omega^2$

From the geometry: $\sin\theta = \frac{r}{l}$

$\Rightarrow T_1 + T_2 = m l \omega^2 \quad \dots (I)$

Resolving forces vertically: — *Comment: Upward dir'n taken as +ve*

$T_1 \cos\theta - T_2 \cos\theta - mg = 0$

From the geometry $\cos\theta = \frac{l/2}{l}$ or $\frac{1}{2}$.

$\Rightarrow T_1 - T_2 = 2mg \quad \dots (II)$

(I) + (II) gives $2T_1 = m(l\omega^2 + 2g)$ — *Comment: ✓✓ for process*

$\Rightarrow T_1 = \frac{m}{2}(l\omega^2 + 2g) \quad \dots (III)$

Substituting (III) into (I) gives

$\frac{m}{2}(l\omega^2 + 2g) + T_2 = m l \omega^2$

$\Rightarrow T_2 = \frac{m}{2}(l\omega^2 - 2g) \dots (IV)$

ii) For the strings to remain taut we require:-

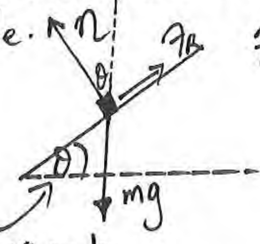
$T_1, T_2 > 0$. Now T_1 is always +ve whereas T_2 has its threshold value when $T_2 = 0$

$\Rightarrow \frac{m}{2}(l\omega^2 - 2g) = 0$

$\Rightarrow \omega = \sqrt{\frac{2g}{l}}$ — *Comment: $\omega > 0$*

So $\omega > \sqrt{\frac{2g}{l}}$ ✓

(c) The lateral thrust between the tyres and the car must be negligible. $F_R \approx 0$.



Resolving forces horizontally & vertically :- — *Comment: Radially inward*

Horizontally: $N \sin\theta = m v^2 / r \quad \dots (I)$

Vertically: $N \cos\theta = mg \quad \dots (II)$

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Question 15 Cont. c) Cont. (I) ÷ (II) gives $\tan\theta = \frac{v^2}{rg}$ $\Rightarrow \theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$, θ acute Now $r = 200\text{m}$; $v = 100 \text{ km/h}$ $g = 9.8 \text{ m/s}^2$ $= 27\frac{7}{9} \text{ m/s}$. $\Rightarrow \theta = \tan^{-1}(0.3936\ldots)$ $\simeq 21.5^\circ$ ∴ banking angle is approx 21.5°. d) (i) $\frac{d}{d\theta}\left(\sin^{n-1}\theta \cdot \cos\theta\right)$ $= \sin^{n-1}\theta \cdot \frac{d}{d\theta}(\cos\theta) + \cos\theta \cdot \frac{d}{d\theta}\left(\sin^{n-1}\theta\right)$ $= -\sin^n\theta + (n-1)\cdot\sin^{n-2}\theta\cdot\cos^2\theta$ $= -\sin^n\theta + (n-1)\cdot\sin^{n-2}\theta\,(1-\sin^2\theta)$ $= -\sin^n\theta + (n-1)\cdot\sin^{n-2}\theta - (n-1)\sin^n\theta$ $= (n-1)\cdot\sin^{n-2}\theta - n\cdot\sin^n\theta$. (ii) From (i) $n\sin^n\theta = (n-1)\cdot\sin^{n-2}\theta - \frac{d}{d\theta}\left(\sin^{n-1}\theta\cdot\cos\theta\right)$ $\therefore \int_0^{\pi/2}\sin^n\theta = \frac{n-1}{n}\cdot\int_0^{\pi/2}\sin^{n-2}\theta - \frac{1}{n}\int_0^{\pi/2}\frac{d}{d\theta}\left(\sin^{n-1}\theta\cdot\cos\theta\right)d\theta$ $\Rightarrow I_n = \frac{n-1}{n}\cdot I_{n-2} - \frac{1}{n}\left[\sin^{n-1}\theta\cdot\cos\theta\right]_0^{\pi/2}$ The latter term evaluates to zero as $\cos\frac{\pi}{2} = 0 = \sin 0$ So $I_n = \left(\frac{n-1}{n}\right)\cdot I_{n-2}$, as required	✓✓ for process. 21.488368.. Product Rule Chain rule ✓✓	(iii) $I_4 = \frac{3}{4}\cdot I_2$ $= \frac{3}{4}\cdot\frac{1}{2}\cdot I_0$ & $I_0 = \int_0^{\pi/2} 1\,d\theta$ or $\pi/2$. $\therefore I_4 = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$. i.e. $I_4 = \frac{3\pi}{16}$ ✓ Question 16: (a) $P(n)$: $n^{n+1} > n(n+1)^{n-1}$; $n \in \mathbb{N} \setminus \{0,1\}$. Test for $n = 2$ $LHS_{P(2)} = 2^3$ or 8 ; $RHS_{P(2)} = 2\times3^1$ or 6 As $8 > 6$, $P(2)$ is true Assume the prop^n is true for some arbitrary counting no. $n = K$, i.e. assume $P(K)$ is true or equivalently $K^{K+1} > K(K+1)^{K-1}$ Goal: Prove the result is necessarily true for $n = K+1$, i.e. $(K+1)^{K+2} > (K+1)(K+2)^K$ when $K^{K+1} > K(K+1)^{K-1}$. i.e. $P(K) \Rightarrow P(K+1)$. See next page	Using Reduction Formula Basis Step. Inductive Hypothesis.

Suggested Solution (s)	Comments
Question 16 Cont. a) cont.	
$LHS_{P(K+1)} = (K+1)^{K+2}$	
$= \dfrac{(K+1)^{K+2}\cdot K^{K+1}}{K^{K+1}}$	
$> \dfrac{(K+1)^{K+2}\cdot K(K+1)^{K-1}}{K^{K+1}}$	by induction hypothesis
$= \dfrac{(K+1)^{2K+1}}{K^{K}}$	
$= \dfrac{\left[(K+1)^2\right]^{K}(K+1)}{K^{K}}$	
$= \left(\dfrac{K^2+2K+1}{K}\right)^{K}(K+1)$	
$= (K+1)\left(K+2+\frac{1}{K}\right)^{K}$	
$> (K+1)(K+2)^{K}$	as $\frac{1}{K} > 0$
$= RHS_{P(K+1)}$.	
So $P(K) \Rightarrow P(K+1)$. That is, if the proposition is true for $n=K$, it is also true for $n=K+1$. As $P(2)$ is true (from basis step), hence $P(3)$ is true and similarly $P(4)$ and hence forth for all remaining counting no.s.	1 mark: Tests the result for $n=2$; 3 marks for correct proof.

Suggested Solution (s)	Comments
(b) i) From $x \geq 0 \Rightarrow 1+x \geq 1 (>0)$	Adding one to both sides
$\Rightarrow (0<)\dfrac{1}{x+1} \leq 1$ upon reciprocation	
So $\dfrac{1}{x+1} \leq 1 \quad \ldots (I)$	
Also, since $x \geq 0$, certainly $x^2 \geq 0$.	
$\Rightarrow -x^2 \leq 0$	mult^n by -1.
$\Rightarrow 1-x^2 \leq 1$ or $(1-x)(1+x) \leq 1$	Adding one to both sides
$\Rightarrow 1-x \leq \dfrac{1}{1+x}$ upon division by $1+x$	$1+x \geq 1 (>0)$
So $1-x \leq \dfrac{1}{1+x} \quad \ldots (II)$	✓✓ for proof.
From (I) & (II) $1-x \leq \dfrac{1}{1+x} \leq 1$, as required.	
ii) Noting $\displaystyle\int_0^{1/n} \frac{dx}{1+x} = \ln(1+1/n)$, then	✓✓ for process
$\displaystyle\int_0^{1/n}(1-x)\,dx \leq \int_0^{1/n}\frac{dx}{1+x} \leq \int_0^{1/n} dx$	
$\Rightarrow \dfrac{1}{n} - \dfrac{1}{2n^2} \leq \ln\left(1+\frac{1}{n}\right) \leq \dfrac{1}{n}$	
$\Rightarrow 1 - \dfrac{1}{2n} \leq n\ln(1+1/n) \leq 1$ upon mult^n by n.	
iii) We have $1-\dfrac{1}{2n} \leq \ln(1+1/n)^n \leq 1$	from (ii) $n\ln(x) = \ln(x^n)$
Exponentiating this inequality we have $e^{1-\frac{1}{2n}} \leq e^{\ln(1+\frac{1}{n})^n} \leq e^1$	e^x is an increasing f^n & so the inequalities are preserved.
$\Rightarrow e^{1-\frac{1}{2n}} \leq \left(1+\frac{1}{n}\right)^n \leq e^1$	
$\Rightarrow \lim\limits_{n\to\infty} e^{1-\frac{1}{2n}} \leq \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n \leq \lim\limits_{n\to\infty} e$	$e^{\ln x} = x$ for $x>0$
$\Rightarrow e^{\lim\limits_{n\to\infty}(1-\frac{1}{2n})} \leq \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n \leq \lim\limits_{n\to\infty} e$	limit properties plus Squeeze theorem (formally)
$\Rightarrow e \leq \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n \leq e$	
& so $\lim\limits_{n\to\infty}(1+1/n)^n = e$, as required	

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Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

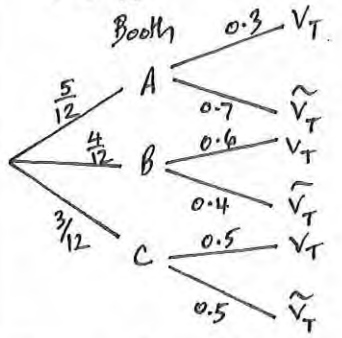
Question 16 cont:

(c) Let A, B & C be the events that the vote is drawn from booth A, B & C respectively. Also, let V_T be the event that the vote is for Mr. Turnbull. We require initially the probability that a randomly selected vote will be for Mr. Turnbull.

$P(V_T) = P(V_T/A)\cdot P(A) + P(V_T/B)\cdot P(B) + P(V_T/C)\cdot P(C)$

$= 0.3 \times \frac{5}{12} + 0.6 \times \frac{4}{12} + 0.5 \times \frac{3}{12}$

$= 0.45$



Comment: Tree diagram useful in deciphering the logic.

For multiple voters, the distribution is essentially binomial with $P(\text{SUCCESS}) = P(S) = 0.45$ & $P(\text{FAILURE}) = P(\tilde{S}) = 0.55$ & n (the no. of trials) $= 10$.

Let X be the discrete random variable representing the no. of votes for Mr. Turnbull from ten ballots. We require $P_r(X \geq 8)$

Comments: 1 mark: Calculates probability of one voter for Mr Turnbull. 2 marks: Mostly correct solution. 3 marks: Correct sol'n.

$X = 0, 1, 2, \ldots, 10$

$P_r(X \geq 8) = P(X=8) + P(X=9) + P(X=10)$

$= {}^{10}C_8 (0.45)^8 (0.55)^2 + {}^{10}C_9 (0.45)^9 (0.55) + (0.45)^{10}$

$\simeq \underline{0.02739} \quad (0.027391839\ldots)$

(d) i) From $(a-b)^2 \geq 0$

$\Rightarrow \quad a^2 + b^2 \geq 2ab \quad \ldots (I)$

Similarly from $(a-c)^2 \geq 0$ & $(b-c)^2 \geq 0$ we obtain:-

$a^2 + c^2 \geq 2ac \quad \ldots (II)$ &

$b^2 + c^2 \geq 2bc \quad \ldots (III)$

Adding (I), (II) & (III) we obtain:-

$2(a^2+b^2+c^2) \geq 2(ab+ac+bc)$

$\Rightarrow \quad \underline{a^2+b^2+c^2 \geq ab+ac+bc} \quad \ldots (IV)$

Now $(a+b+c)^2 = a^2+b^2+c^2 + 2(ab+ac+bc)$

$\Rightarrow (a+b+c)^2 \geq ab+ac+bc + 2(ab+ac+bc)$ using (IV)

& the result follows.

Comment: Algebraic Expansion & collection of like terms.

(ii) From (IV) above:-

$a^2+b^2+c^2 \geq ab+ac+bc.$

Let $a = xy$, $b = xz$ & $c = yz$ in this inequality & the result follows. LHS $= x^2y^2 + x^2z^2 + y^2z^2$ & RHS $= x^2yz + xy^2z + xyz^2$ or $xyz(x+y+z)$.

Comment: Otherwise approach

Cont. over

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Question 16 Cont. d) ii) Cont. From $(a+b+c)^2 \geq 3(ab+ac+bc)$, let $a = xy$, $b = xz$, $c = yz$. Then inequality becomes :- $(xy+xz+yz)^2 \geq 3(x^2yz + xy^2z + xyz^2)$ $LHS = x^2y^2 + x^2z^2 + y^2z^2 + 2(x^2yz + xy^2z + xyz^2)$ i.e. $x^2y^2 + x^2z^2 + y^2z^2 + 2xyz(x+y+z)$ $RHS = 3xyz(x+y+z)$. So $x^2y^2 + x^2z^2 + y^2z^2 + 2xyz(x+y+z) \geq 3xyz(x+y+z)$ whence $x^2y^2 + x^2z^2 + y^2z^2 \geq xyz(x+y+z)$ as required. τὸ τέλος	Hence approach ✓✓ for process		